

Vorticity Knot in Two-component Bose-Einstein Condensates

Y. M. Cho*

*School of Physics, College of Natural Sciences,
Seoul National University, Seoul 151-742, Korea
and*

*C.N.Yang Institute for Theoretical Physics,
State University of New York,
Stony Brook, NY 11790, USA*

We demonstrate the existence of the helical vortex solution in two-component Bose-Einstein condensates which can be identified as a twisted vorticity flux. Based on this we argue that the recently proposed knot in two-component Bose-Einstein condensates can be interpreted as a vorticity knot, a vortex ring made of the helical vortex. This picture shows that the knot is made of two quantized vorticity fluxes linked together, whose topology $\pi_3(S^2)$ is fixed by the linking number of two vorticity fluxes. Due to the helical structure the knot has both topological and dynamical stability. We estimate the energy of the lightest knot to be about 3×10^{-3} eV.

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The topological objects, in particular finite energy topological objects, have played an important role in physics [1, 2]. In Bose-Einstein condensates (BEC) the best known topological objects are the vortices, which have been widely studied in the literature. Theoretically these vortices have successfully been described by the Gross-Pitaevskii Lagrangian. On the other hand, the recent advent of multi-component BEC (in particular the spin-1/2 condensate of ^{87}Rb atoms) has widely opened a new opportunity for us to study novel topological objects which can not be realized in ordinary (one-component) BEC [3, 4]. This is because the multi-component BEC naturally allows a non-Abelian structure which accommodates a non-trivial topological objects, in particular a topological knot which is very similar to the knot in Skyrme theory [5, 6].

Indeed recently many authors have proposed the existence of a knot in Gross-Pitaevskii theory of two-component BEC [7, 8, 9]. *The purpose of this report is to show that this knot is nothing but a vorticity knot which is made of two vorticity fluxes linked together. Furthermore, we show that the knot is topological, whose topology $\pi_3(S^2)$ is fixed by the Chern-Simon index of the velocity potential of the condensate.* To show this we first present a helical vortex solution in two-component BEC which is periodic in z -coordinate, and construct a helical vortex ring by bending it and smoothly connecting two periodic ends together. We show that this vortex ring becomes the vorticity knot whose quantum number is fixed by the Chern-Simon index of the velocity potential, which describes the linking number of two vorticity fluxes.

This picture tells that the knot has both topological and dynamical stability. The topological stability follows

from the fact that two linked vorticity fluxes can not be disconnected by any smooth deformation of the field configuration. The dynamical stability follows from the fact that the knot necessarily has a net velocity flux along the knot, and thus a non-vanishing angular momentum around the knot. This creates a repulsive stabilizing force against the collapse of the knot. This provides the dynamical stability of the knot.

The knot that we discuss here are very similar to the knot in Skyrme theory [6, 10, 11]. Just as the knot in Skyrme theory is a vortex ring made of the helical magnetic vortex, our knot here is a vortex ring made of the helical vorticity vortex. So it is crucial that we have the helical vortex to demonstrate the existence of the vorticity knot in two-component BEC.

To construct the desired vortex solution let the two-component BEC be a complex doublet $\phi = (\phi_1, \phi_2)$, and consider the Lagrangian

$$\mathcal{L} = i\frac{\hbar}{2}\phi^\dagger\partial_t\phi - \frac{\hbar^2}{2M}|\partial_i\phi|^2 + \mu_1\phi_1^\dagger\phi_1 + \mu_2\phi_2^\dagger\phi_2 - \frac{\lambda_{11}}{2}(\phi_1^\dagger\phi_1)^2 - \lambda_{12}(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - \frac{\lambda_{22}}{2}(\phi_2^\dagger\phi_2)^2, \quad (1)$$

where μ_i are the quadratic coupling constants and λ_{ij} are the quartic coupling constants which are determined by the scattering lengths a_{ij}

$$\lambda_{ij} = \frac{4\pi\hbar^2}{M}a_{ij}. \quad (2)$$

This is an obvious generalization of one-component Gross-Pitaevskii Lagrangian to the two-component BEC. Notice that here we have neglected the trapping potential, because we are assuming that the range of the trapping potential is much larger than the size of topological objects we are interested in.

*Electronic address: ymcho@yongmin.snu.ac.kr

Clearly the Lagrangian has a global $U(1) \times U(1)$ symmetry. But one could simplify it because experimentally the scattering lengths often have almost the same value. For example, for the spin 1/2 condensate of ^{87}Rb atoms, all a_{ij} are about 5.5 nm and differ by only about 3 % or so [3, 4]. In this case one may safely assume $\lambda_{11} \simeq \lambda_{12} \simeq \lambda_{22} \simeq \bar{\lambda}$. With this the Lagrangian is written as

$$\mathcal{L} = i\frac{\hbar}{2}\phi^\dagger \partial_t \phi - \frac{\hbar^2}{2M}|\partial_i \phi|^2 - \frac{\bar{\lambda}}{2}(\phi^\dagger \phi - \frac{\mu}{\lambda})^2 - \delta\mu \phi_1^\dagger \phi_2, \quad (3)$$

where $\mu = \mu_1$ and $\delta\mu = \mu_1 - \mu_2$. Notice that the Lagrangian has a global $U(2)$ symmetry when $\delta\mu = 0$. So the $\delta\mu$ interaction is the symmetry breaking term which breaks the global $U(2)$ symmetry to $U(1) \times U(1)$. This means that even when $\delta\mu \neq 0$ the Lagrangian has an approximate $U(2)$ symmetry. Physically $\delta\mu$ can be viewed to represent the difference of the chemical potentials between ϕ_1 and ϕ_2 , so that it does not vanish when the chemical potentials are different.

With

$$\phi = \frac{1}{\sqrt{2}}\rho\zeta, \quad (\zeta^\dagger \zeta = 1) \quad (4)$$

the Lagrangian (3) gives the following Hamiltonian in the static limit (in the natural unit $c = \hbar = 1$),

$$\mathcal{H} = \frac{1}{2}(\partial_i \rho)^2 + \frac{1}{2}\rho^2|\partial_i \zeta|^2 + \frac{\lambda}{8}(\rho^2 - \frac{2\mu}{\lambda})^2 + \frac{\delta\mu^2}{2}\rho^2\zeta_2^*\zeta_2, \quad (5)$$

where $\lambda = 4M^2\bar{\lambda}$, $\delta\mu^2 = 2M\delta\mu$, $\rho_0^2 = 4\mu M/\lambda$, and we have normalized ρ to $(\sqrt{2M}/\hbar)\rho$. The Hamiltonian (5) can be expressed as

$$\mathcal{H} = \lambda\rho_0^4 \left\{ \frac{1}{2}(\hat{\partial}_i \hat{\rho})^2 + \frac{1}{2}\hat{\rho}^2|\hat{\partial}_i \zeta|^2 + \frac{1}{8}(\hat{\rho}^2 - 1)^2 + \frac{\delta\mu}{4\mu}\hat{\rho}^2\zeta_2^*\zeta_2 \right\}, \quad (6)$$

where $\hat{\rho} = \rho/\rho_0$ and $\hat{\partial}_i = \partial_i/\sqrt{\lambda}\rho_0$. This tells that the physical unit of the Hamiltonian is $\lambda\rho_0^4$, and the physical scale κ of the coordinates is $1/\sqrt{\lambda}\rho_0$. This is comparable to the correlation length $\bar{\xi} = 1/\sqrt{2\mu M}$. Indeed we have $\kappa = \bar{\xi}/\sqrt{2}$.

From the Hamiltonian we have

$$\begin{aligned} \partial^2 \rho - |\partial_i \zeta|^2 \rho &= \left(\frac{\lambda}{2}(\rho^2 - \rho_0^2) + \delta\mu^2(\zeta_2^*\zeta_2) \right) \rho, \\ \left\{ (\partial^2 - \zeta^\dagger \partial^2 \zeta) + 2\frac{\partial_i \rho}{\rho}(\partial_i - \zeta^\dagger \partial_i \zeta) + \delta\mu^2(\zeta_2^*\zeta_2) \right\} \zeta_1 &= 0, \\ \left\{ (\partial^2 - \zeta^\dagger \partial^2 \zeta) + 2\frac{\partial_i \rho}{\rho}(\partial_i - \zeta^\dagger \partial_i \zeta) - \delta\mu^2(\zeta_1^*\zeta_1) \right\} \zeta_2 &= 0. \end{aligned} \quad (7)$$

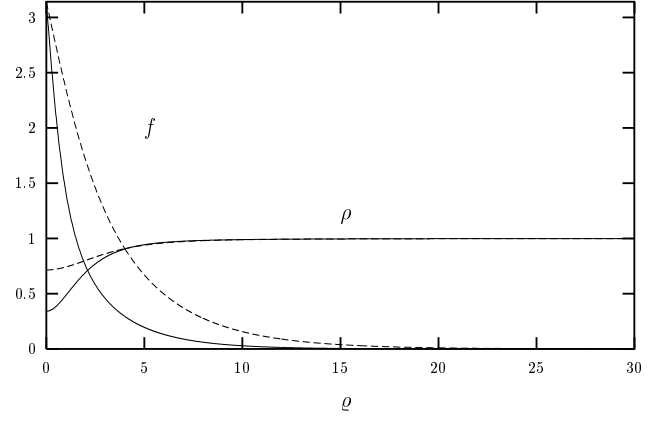


FIG. 1: The helical vortex in the Gross-Pitaevskii theory of two-component BEC. Here we have put $m = 1, m' = -1, n = 1, n' = 0, k = 0.25/\kappa$, and ϱ is in the unit of κ . Dashed and solid lines correspond to $\delta\mu/\mu = 0$, and 0.1 respectively.

To obtain the vortex solution, we choose the ansatz

$$\begin{aligned} \rho &= \rho(\varrho), \\ \zeta &= \exp(-i\gamma) \xi, \quad \gamma = n'\varphi + m'kz, \\ \xi &= \begin{pmatrix} \cos \frac{f(\varrho)}{2} \exp(-in\varphi - imkz) \\ \sin \frac{f(\varrho)}{2} \end{pmatrix}. \end{aligned} \quad (8)$$

Now, with $n' = 0$ and $m' = -m$ (7) is reduced to

$$\begin{aligned} \ddot{\rho} + \frac{1}{\varrho}\dot{\rho} - \left(\frac{1}{4}\dot{f}^2 + \frac{n^2}{\varrho^2} \right. \\ \left. - \left(\frac{n^2}{\varrho^2} - m^2k^2 - \delta\mu^2 \right) \sin^2 \frac{f}{2} \right) \rho &= \frac{\lambda}{2}(\rho^2 - \rho_0^2)\rho, \\ \ddot{f} + \left(\frac{1}{\varrho} + 2\frac{\dot{\rho}}{\rho} \right) \dot{f} + \left(\frac{n^2}{\varrho^2} - m^2k^2 - \delta\mu^2 \right) \sin f &= 0. \end{aligned} \quad (9)$$

So with the boundary condition

$$\rho'(0) = 0, \quad \rho(\infty) = \rho_0, \quad f(0) = \pi, \quad f(\infty) = 0, \quad (10)$$

we can solve (9). With $m = n = 1$ we obtain the twisted vortex solution shown in Fig. 1.

The untwisted non-Abelian vortex solution has been discussed before [9], but the twisted vortex solution here is new. Notice that when $\delta\mu^2 = 0$, there is no untwisted vortex solution because in this case the vortex size become infinite. But remarkably the helical vortex exists even when $\delta\mu^2 = 0$. This is because the twisting reduces the size of vortex tube.

In Skyrme theory the helical vortex is interpreted as a twisted magnetic vortex whose flux is quantized [6, 11]. Now we show that the above vortex is a twisted vorticity vortex. To see this notice that the non-Abelian structure of the vortex is represented by the doublet ζ . Moreover,

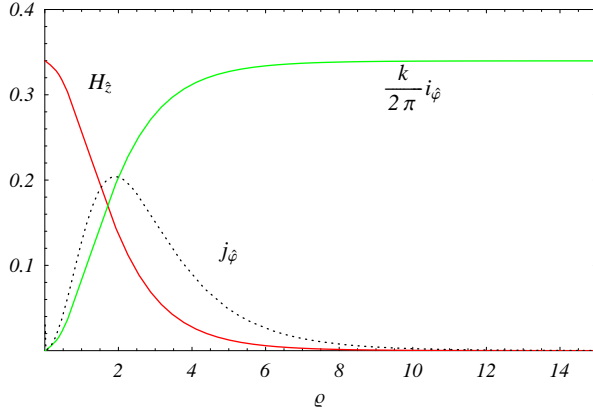


FIG. 2: The supercurrent $i_{\hat{\phi}}$ (in one period section in z -coordinate) and corresponding magnetic field $H_{\hat{z}}$ circulating around the cylinder of radius ϱ of the helical vortex in two-component BEC. Here $m = 1, m' = -1, n = 1, n' = 0, k = 0.25/\kappa$, and ϱ is in the unit of κ . The current density $j_{\hat{\phi}}$ is represented by the dotted line.

the velocity field of the doublet is given by [5]

$$V_{\mu} = i\zeta^{\dagger}\partial_{\mu}\zeta = i\xi^{\dagger}\partial_{\mu}\xi + \partial_{\mu}\gamma \\ = \frac{1}{2}(\cos f(\varrho) + 1)(n\partial_{\mu}\varphi + mk\partial_{\mu}z) + \partial_{\mu}\gamma, \quad (11)$$

which generates the vorticity

$$V_{\mu\nu} = \partial_{\mu}V_{\nu} - \partial_{\nu}V_{\mu} = i(\partial_{\mu}\xi^{\dagger}\partial_{\nu}\xi - \partial_{\nu}\xi^{\dagger}\partial_{\mu}\xi) \\ = -\frac{\dot{f}}{2}\sin f \left(n(\partial_{\mu}\varrho\partial_{\nu}\varphi - \partial_{\nu}\varrho\partial_{\mu}\varphi) \right. \\ \left. + mk(\partial_{\mu}\varrho\partial_{\nu}z - \partial_{\nu}\varrho\partial_{\mu}z) \right). \quad (12)$$

This has two vorticity fluxes, $\phi_{\hat{z}}$ along the z -axis

$$\phi_{\hat{z}} = \int V_{\hat{\phi}\hat{\phi}} d\varrho d\varphi = 2\pi n, \quad (13)$$

and $\phi_{\hat{\phi}}$ around the the z -axis (in one period section from $z = 0$ to $z = 2\pi/k$)

$$\phi_{\hat{\phi}} = \int_0^{2\pi/k} V_{\hat{z}\hat{z}} d\varrho dz = -2\pi m. \quad (14)$$

Obviously they are quantized. As importantly they are linked together, and have the linking number mn .

Furthermore, just as in Skyrme theory, these fluxes can be viewed to originate from the helical supercurrent which confines them with a built-in Meissner effect [11]

$$j_{\mu} = \partial_{\nu}V_{\mu\nu} \\ = \sin f \left[n \left(\ddot{f} + \frac{\cos f}{\sin f} \dot{f}^2 - \frac{1}{\varrho} \dot{f} \right) \partial_{\mu}\varphi \right. \\ \left. + mk \left(\ddot{f} + \frac{\cos f}{\sin f} \dot{f}^2 + \frac{1}{\varrho} \dot{f} \right) \partial_{\mu}z \right]. \quad (15)$$

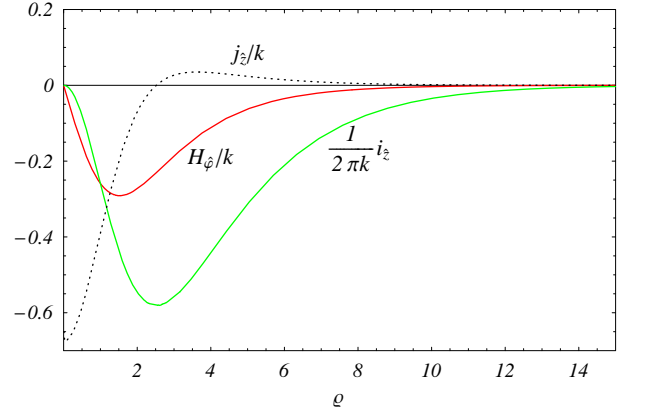


FIG. 3: The supercurrent $i_{\hat{z}}$ and corresponding magnetic field $H_{\hat{\phi}}$ flowing through the disk of radius ϱ of the helical vortex in two-component BEC. Here $m = 1, m' = -1, n = 1, n' = 0, k = 0.25/\kappa$, and ϱ is in the unit of κ . The current density $j_{\hat{z}}$ is represented by the dotted line.

This produces the supercurrents $i_{\hat{\phi}}$ (in one period section from $z = 0$ to $z = 2\pi/k$) around the z -axis

$$i_{\hat{\phi}} = \frac{2\pi n}{k} \frac{\sin f}{\varrho} \dot{f} \Big|_{\varrho=0}^{\varrho=\infty}, \quad (16)$$

and $i_{\hat{z}}$ along the z -axis

$$i_{\hat{z}} = 2\pi mk \varrho \dot{f} \sin f \Big|_{\varrho=0}^{\varrho=\infty}. \quad (17)$$

The vorticity fluxes and the corresponding supercurrents are shown in Fig. 2 and Fig. 3. This is strikingly similar to what we find in the magnetic vortex in Skyrme theory [11]. This tells that the helical vortex is nothing but the twisted vorticity flux confined along the z -axis by the velocity current, whose flux is quantized due to the topological reason. We emphasize that this interpretation holds even when the $\delta\mu^2$ is not zero.

We can estimate the energy of the helical vortex. For ^{87}Rb we have

$$M \simeq 8.1 \times 10^{10} \text{ eV}, \quad \bar{\lambda} \simeq 1.68 \times 10^{-7} (nm)^2, \\ \mu \simeq 3.3 \times 10^{-12} \text{ eV}, \quad \delta\mu \simeq 0.1 \mu. \quad (18)$$

So, with $m = n = 1, m' = -1, n' = 0$ and $k = 0.25/\kappa$, we find numerically that the energy per one periodic section (from $z = 0$ to $z = 2\pi/k$) is given by

$$E \simeq 270.987 \frac{\rho_0^2}{\sqrt{\lambda} \mu} \simeq 1492.4 \rho_0 \\ \simeq 2.29 \times 10^{-3} \text{ eV}. \quad (19)$$

As we will see later, the lightest knot could have an energy comparable to this energy.

Notice that the vorticity (12) is completely fixed by the CP^1 field ξ , because it does not depend on the $U(1)$

phase γ of ζ . Moreover ξ naturally defines a mapping from the compactified xy -plane S^2 to the target space S^2 . This means that our vortex has exactly the same topological origin as the baby skyrmion in Skyrme theory, but now the topological quantum number is expressed by $\pi_2(S^2)$ of the condensate ξ ,

$$q = -\frac{i}{4\pi} \int \epsilon_{ij} \partial_i \xi^\dagger \partial_j \xi d^2x = n. \quad (20)$$

This clarifies the topological origin of the non-Abelian vortex in two-component BEC.

The helical vortex will become unstable unless the periodicity condition is enforced by hand. But just as in Skyrme theory we can make it a stable knot by smoothly connecting two periodic ends. In this knot the periodicity condition is automatically guaranteed, and the very twist which causes the instability of the helical vortex now ensures the stability of the knot. This is so because dynamically the momentum mk along the z -axis created by the twist now generates a velocity current and thus a net angular momentum which provides the centrifugal repulsive force preventing the knot to collapse.

Furthermore, this dynamical stability of the knot is now backed up by the topological stability. This is because mathematically the doublet ξ , after forming a knot, acquires a non-trivial topology $\pi_3(S^2)$. And the knot quantum number is given by the Chern-Simon index of the velocity potential,

$$\begin{aligned} Q &= -\frac{1}{4\pi^2} \int \epsilon_{ijk} \xi^\dagger \partial_i \xi (\partial_j \xi^\dagger \partial_k \xi) d^3x \\ &= \frac{1}{16\pi^2} \int \epsilon_{ijk} V_i V_j d^3x = mn. \end{aligned} \quad (21)$$

This is precisely the linking number of two vorticity fluxes. As importantly, this is formally identical to the knot quantum number in Skyrme theory [6, 10, 11]. This

assures the topological stability of the knot, because two fluxes linked together can not be disconnected by any smooth deformation of the field configuration.

We can estimate the energy of the knot, noticing that the radius of the lowest energy vortex ring is about four times the vortex tube size [12]. This suggests that the lightest knot has the energy comparable to the energy of the lightest helical vortex in one periodic section with $k \simeq 1/4\kappa$. So the lightest knot in ^{87}Rb is expected to have the energy of the order of $3 \times 10^{-3} \text{ eV}$.

The existence of a knot in Gross-Pitaevskii theory of two-component BEC has been proposed by several authors [7, 8, 9]. In this paper we have clarified the physical meaning of the knot. Just as the knot in Skyrme theory is a twisted magnetic flux ring, this knot is a twisted vorticity flux ring. It has a topological quantum number given by the Chern-Simon index of the velocity potential of the condensate, and enjoys both topological and dynamical stability.

What is remarkable is that this knot is almost identical to the knot in the gauge theory of two-component BEC that we proposed recently [5]. Both are vorticity knots whose topology is identical. This implies that we have two competing theories of two-component BEC, the Gross-Pitaevskii theory and the recently proposed gauge theory, which can describe the knot.

Constructing the knot might not be simple, but might have already been done [4, 13]. Identifying it as a vorticity knot, however, may be a challenging task. A detailed discussion on the subject will be published elsewhere [14].

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- [1] A. Abrikosov, Sov. Phys. JETP **5**, 1174 (1957).
 - [2] T. H. R. Skyrme, Proc. Roy. Soc. (London) **260**, 127 (1961); **262**, 237 (1961); Nucl. Phys. **31**, 556 (1962).
 - [3] C. Myatt *at al.*, Phys. Rev. Lett. **78**, 586 (1997); J. Stenger *at al.*, Nature **396**, 345 (1998); M. Matthews *at al.*, Phys. Rev. Lett. **83** 2498 (1999); K. Madison *at al.*, Phys. Rev. Lett. **84**, 806 (2000).
 - [4] D. Hall *at al.*, Phys. Rev. Lett. **81**, 1536 (1998); C. Law *at al.*, Phys. Rev. Lett. **81**, 5215 (1998); J. Williams and M. Holland, Nature **401**, 568 (1999).
 - [5] Y. M. Cho, cond-mat/0112325; cond-mat/0308182.
 - [6] Y. M. Cho, Phys. Rev. Lett. **87**, 252001 (2001).
 - [7] J. Ruostekoski and J. Anglin, Phys. Rev. Lett. **86**, 3934 (2001); H. Stoof *at al.*, Phys. Rev. Lett. **87**, 120407 (2001); U. Al Khawaja and H. Stoof, Nature **411**, 818 (2001).
 - [8] R. Battye, N. Cooper, and P. Sutcliffe, Phys. Rev. Lett. **88**, 080401 (2002); C. Savage and J. Ruostekoski, Phys. Rev. Lett. **91**, 010403 (2003).
 - [9] M. Metlitski and A. Zhitnitsky, cond-mat/0307559.
 - [10] L. Faddeev and A. Niemi, Nature **387**, 58 (1997); J. Gladikowski and M. Hellmund, Phys. Rev. **D56**, 5194 (1997); R. Battye and P. Sutcliffe, Phys. Rev. Lett. **81**, 4798 (1998).
 - [11] Y. M. Cho, nucl-th/0309051, Phys. Lett. **B**, in press; hep-th/0404181.
 - [12] Y. M. Cho, cond-mat/0112498; cond-mat/0311201.
 - [13] P. Anderson *at al.*, Phys. Rev. Lett. **86**, 2926 (2001).
 - [14] Y. M. Cho and H. J. Khim, in preparation.